Electrodynamic Casimir effect in a medium-filled wedge

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We re-examine the electrodynamic Casimir effect in a wedge defined by two perfect conductors making dihedral angle $\alpha=\pi/p$. This system is analogous to the system defined by a cosmic string. We consider the wedge region as filled with an azimuthally symmetric material, with permittivity and permeability ε_1, μ_1 for distance from the axis $r<a$, and ε_2, μ_2 for $r>a$. The results are closely related to those for a circular-cylindrical geometry, but with noninteger azimuthal quantum number mp. Apart from a zero-mode divergence, which may be removed by choosing periodic boundary conditions on the wedge, and may be made finite if dispersion is included, we obtain finite results for the free energy corresponding to changes in a for the case when the speed of light is the same inside and outside the radius a, and for weak coupling, $|\varepsilon_1-\varepsilon_2|\ll 1$, for purely dielectric media. We also consider the radiation produced by the sudden appearance of an infinite cosmic string, situated along the cusp line of the pre-existing wedge.

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I. INTRODUCTION

Quantum field theory in the wedge geometry continues to attract interest, especially in connection with the Casimir effect. Usually it is assumed that the interior region of the wedge is a vacuum, and that the two plane surfaces $\theta=0$ and $\theta=\alpha$ (α denotes the opening angle) are perfectly conducting. The coordinate system is conventionally oriented such that the z axis coincides with the singularity axis, i.e., the intersection line for the planes. For an introduction to the wedge model one may consult the book of Mostepanenko and Trunov [1].

The Casimir energy and stress in a wedge geometry was approached already in the 1970s [2,3]. Since that time, various embodiments of the wedge with perfectly conducting walls have been treated by Brevik and co-workers [4–6] and others [7]. More recently a wedge intercut by a cylindrical shell was considered by Nesterenko and collaborators, first for a semicircular wedge [8], then for arbitrary dihedral angle [9]. Local Casimir stresses were examined by Saharian and co-workers [10–12]. Rosa and collaborators studied the interaction of an atom with a wedge [13,14], the situation under which the closely related Casimir-Polder force was investigated by Sukenik *et al.* some years ago [15]. That interaction was first worked out by Barton [16].

One reason for the interest in the wedge geometry is the similarity with the formalism encountered in Casimir theory of systems having circular symmetry. This applies to the case of a perfectly conducting circular boundary [17–20], as well as to the case of a dielectric circular boundary [21–25]. Another reason for studying the wedge is the analogy—at least in a formal sense—with the theory of a cosmic string (cf., for instance, Ref. [26] or [4]). Let us briefly elaborate on the last-mentioned point. The line element outside a cosmic string is, in standard notation,

$$ds^2=-dt^2+dr^2+(1-4GM)^2r^2d\theta^2+dz^2, \tag{1.1}$$

where G is the gravitational constant and M the string mass per unit length. This is the geometry of locally flat space, with a deficit angle $\Phi=8\pi GM$ being removed. Let us introduce the symbols β and p by

$$\beta=(1-4GM)^{-1}=(1-\Phi/2\pi)^{-1}, \tag{1.2a}$$

$$p=\pi/\alpha. \tag{1.2b}$$

Now comparing the electromagnetic energy-momentum tensor outside the string [27]

$$\langle T_{\mu\nu}\rangle=\frac{1}{720\pi^2r^4}(\beta^2+11)(\beta^2-1)\mathrm{diag}(1,-3,1,1) \tag{1.3}$$

with the electromagnetic energy-momentum in the wedge [2–4]

$$\langle T_{\mu\nu}\rangle=\frac{1}{720\pi^2r^4}\left(\frac{\pi^2}{\alpha^2}+11\right)\left(\frac{\pi^2}{\alpha^2}-1\right)\mathrm{diag}(1,-3,1,1), \tag{1.4}$$

we see that β corresponds to p. Hence the deficit angle Φ corresponds to $2\pi-2\alpha$. We shall return to this analogy later. Note that the stress tensor diverges at $r=0$, which makes the definition of a total Casimir energy in these configurations problematic. (Possible solutions to this problem were offered by Khusnutdinov and Bordag [28].)

A particular variant of the wedge model occurs if we introduce a cylindrical boundary of radius a in the cavity. The

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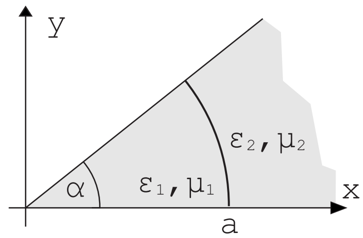

FIG. 1. The geometry considered in Secs. II and III. There is a cylindrical perfectly conducting shell at radius a. In these sections the indices of refraction are equal, $n^2=\epsilon_1\mu_1=\epsilon_2\mu_2$.

situation is sketched in Fig. 1. This model has been studied in particular by Nesterenko *et al.* and by Saharian *et al.*; cf. Refs. [8–12] with a wealth of further references therein. (For example, the fermionic situation for the circular case was discussed by Bezzera de Mello *et al.* [29].) The model can be looked upon as being intermediate between that of a conventional wedge, and an optical fiber. And that brings us to the main theme of the present paper, namely to study the situation of Fig. 1 in the presence of a dielectric medium, both in the interior $r<a$ as well as in the exterior, $r>a$. We designate the two regions by indices 1 and 2. Thus in the interior the refractive index is $n_1=\sqrt{\epsilon_1\mu_1}$ with ϵ_1 and μ_1 being the permittivity and the permeability, whereas in the exterior we have analogously $n_2=\sqrt{\epsilon_2\mu_2}$. We take all material quantities ϵ_1,μ_1 and ϵ_2,μ_2 to be constant and nondispersive. The special case when the circular boundary is perfectly conducting is included in the general situation when there is simply a dielectric/diamagnetic boundary. The plane surfaces $\theta=0$ and $\theta=\alpha$ are taken to be perfectly conducting, as usual.

We begin in Sec. II by considering the Fourier decomposition of the TE and TM modes when the circular boundary is perfectly conducting. This is the simplest case. Then we move on to give an expression for the Casimir energy. The case of a dielectric/diamagnetic boundary is considered thereafter (Fig. 2). The results for the wedge are in general divergent, not because of the divergence associated with the apex of the wedge, which does not contribute to the outward stress on the circular arc, but because of the corners where the arc meets the sides of the wedge. This divergence may be isolated in the azimuthal zero modes, independent of the angular coordinates. We propose isolation and removal of this divergence; alternatively, if the perfectly conducting boundaries at $\theta=0,\alpha$ are replaced by periodic boundary conditions, these divergences disappear. When either of these devices are employed, we obtain numerical results for the resulting finite

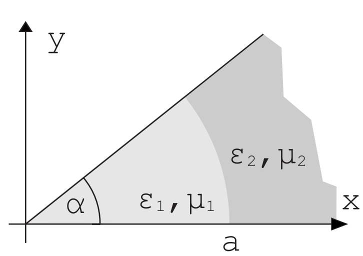

FIG. 2. The wedge with a dielectric/diamagnetic boundary at $r=a$. In Sec. IV we will allow $n_1\neq n_2$.

Casimir energy, referring to the boundary between the two regions, $r<a$ and $r>a$, both for weak and strong coupling. Finally, we exploit the analogy with a cosmic string to calculate, via the Bogoliubov transformation, the production of electromagnetic energy associated with a "sudden" creation of the full wedge situation, as compared with the initial case of a single-medium-filled wedge.

II. ZERO-POINT ENERGY IN THE INTERIOR REGION—PERFECTLY CONDUCTING ARC

As mentioned, we consider an isotropic and homogeneous medium with permittivity ϵ_1 and permeability μ_1 enclosed within a wedge region limited by the conducting plane surfaces $\theta=0$ and $\theta=\alpha$ $(\leq 2\pi)$. In an xy plane, the cusp is situated at the origin. We use cylindrical coordinates (r,θ,z). We employ Heaviside-Lorentz units, and put $\hbar$ and c equal to unity.

Assume, to begin with, that the wedge is closed by a perfectly conducting singular arc at $r=a$. We write down the fundamental modes for stationary electromagnetic modes in the interior wedge, by invoking the expansions given in Ref. [30],

$$E_r=\sum_{m=1}^{\infty}\left[-\frac{2k}{\lambda_1}J'_{mp}(\lambda_1 r)a_m^i-\frac{2i\mu_1\omega mp}{\lambda_1^2 r}J_{mp}(\lambda_1 r)b_m^i\right]\times F_0\sin mp\theta, \tag{2.1a}$$

$$E_\theta=-\sum_{m=0}^{\infty}\left[\frac{2kmp}{\lambda_1^2 r}J_{mp}(\lambda_1 r)a_m^i+\frac{2i\mu_1\omega}{\lambda_1}J'_{mp}(\lambda_1 r)b_m^i\right]\times F_0\cos mp\theta, \tag{2.1b}$$

$$E_z=2i\sum_{m=1}^{\infty}J_{mp}(\lambda_1 r)a_m^iF_0\sin mp\theta, \tag{2.1c}$$

$$H_r=\sum_{m=0}^{\infty}\left[\frac{2mp\epsilon_1\omega}{\lambda_1^2 r}J_{mp}(\lambda_1 r)a_m^i+\frac{2ik}{\lambda_1}J'_{mp}(\lambda_1 r)b_m^i\right]F_0\cos mp\theta, \tag{2.1d}$$

$$H_\theta=-\sum_{m=1}^{\infty}\left[\frac{2\epsilon_1\omega}{\lambda_1}J'_{mp}(\lambda_1 r)a_m^i+\frac{2ikmp}{\lambda_1^2 r}J_{mp}(\lambda_1 r)b_m^i\right]\times F_0\sin mp\theta, \tag{2.1e}$$

$$H_z=2\sum_{m=0}^{\infty}J_{mp}(\lambda_1 r)b_m^iF_0\cos mp\theta. \tag{2.1f}$$

Here k is the axial wave number and λ_1 is the transverse wave number given by

$$\lambda_1^2=n_1^2\omega^2-k^2. \tag{2.2}$$

The J_{mp}'s are ordinary Bessel functions of order mp, which are finite at the origin for $mp\geq 0$, for p nonintegral, while

$$F_0 = \exp(ikz - i\omega t) \tag{2.3}$$

is the $m=0$ version of the more general quantity $F_m = \exp(imp\theta + ikz - i\omega t)$. Expressions (2.1a), (2.1b), (2.1c), (2.1d), (2.1e), and (2.1f) satisfy the electromagnetic boundary conditions on the surfaces $\theta=0$ and $\theta=\alpha$ automatically, for arbitrary values of the coefficients a_m and b_m. The i superscript on the coefficient refers to the interior region. The a_m modes and the b_m modes are independent of each other.

Because of the closure of the region at $r=a$ the problem becomes an eigenvalue problem. Only discrete values of the transverse wave number λ_1 can occur. Let us distinguish between the two kinds of modes:

(i) TM polarization (the a_m^i modes), which correspond to

$$J_{mp}(\lambda_1 a) = 0. \tag{2.4}$$

We denote the roots by $j_{mp,s}$, where $s=1,2,3\ldots$. For a given value of the axial wave number k the energy eigenvalues are accordingly

$$\omega_{msk}^{\mathrm{TM}} = \frac{1}{n_1 a}\sqrt{j_{mp,s}^2 + k^2a^2}, \quad m \geq 1, s \geq 1. \tag{2.5}$$

(ii) TE polarization (the b_m^i modes), which correspond to zeroes of J'_{mp},

$$\omega_{msk}^{\mathrm{TE}} = \frac{1}{n_1 a}\sqrt{(j'_{mp,s})^2 + k^2a^2}, \quad m \geq 0, s \geq 1. \tag{2.6}$$

The interior zero-point energy per unit length is

$$\mathcal{E}^{\mathrm{int}} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{s=1}^{\infty}\left[\omega_{0sk}^{\mathrm{TE}} + \sum_{m=1}^{\infty}(\omega_{msk}^{\mathrm{TM}} + \omega_{msk}^{\mathrm{TE}})\right]. \tag{2.7}$$

We here include the zero-point energy associated with the azimuthally symmetric TE mode, although there is no such TM mode.

To simplify the formalism somewhat, we introduce the symbol $\mathcal{E}_m^{\mathrm{int}}$,

$$\mathcal{E}_m^{\mathrm{int}} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{s=1}^{\infty}(\omega_{msk}^{\mathrm{TM}} + \omega_{msk}^{\mathrm{TE}}). \tag{2.8}$$

For $m=0$ only the TE mode is to be included. We now make use of the argument principle. Any meromorphic function $g(\omega)$ of a complex variable ω satisfies the equation

$$\frac{1}{2\pi i}\oint \omega\frac{d}{d\omega}\ln g(\omega)d\omega = \sum \omega_{\mathrm{zeros}} - \sum \omega_{\mathrm{poles}}, \tag{2.9}$$

where ω_{zeros} are the zeros and ω_{poles} the poles of $g(\omega)$ lying inside the integration contour. The contour is chosen to be a large semicircle in the right half-plane with radius R, closed by a straight line along the imaginary z axis from $\omega=iR$ to $\omega=-iR$. A general advantage of this method is that the multiplicities of zeros as well as for poles are automatically taken care of.

In the present case it is evident that $g(\omega)$ can be chosen as the product of J_{mp} and J'_{mp}. There are no poles involved, and the contribution of the large semicircle goes to zero when $R\to\infty$. Thus we obtain

$$\mathcal{E}_m^{\mathrm{int}} = \frac{1}{2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{i\infty}^{-i\infty}d\omega\omega \times \frac{d}{d\omega}\ln[J_{mp}(\lambda_1 a)J'_{mp}(\lambda_1 a)]. \tag{2.10}$$

In the second integral, k and ω are to be regarded as independent variables in $\lambda_1=\lambda_1(k,\omega)$. We now introduce the imaginary frequency ζ via $\omega\to i\zeta$, whereby

$$\lambda_1 = \sqrt{n_1^2\omega^2 - k^2} \to \sqrt{-(n_1^2\zeta^2 + k^2)} \equiv i\kappa_1. \tag{2.11}$$

We thus encounter Bessel functions of imaginary arguments, $J_{mp}(ix)$, with $x=\kappa_1 a$, $m\geq 0$. Introducing the modified Bessel function $I_\nu(x)$ via $J_\nu(ix)=i^\nu I_\nu(x)$ for arbitrary order ν we get

$$\mathcal{E}_m^{\mathrm{int}} = -\frac{1}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\int_{-\infty}^{\infty}d\zeta\zeta\frac{d}{d\zeta}\ln[I_{mp}(x)I'_{mp}(x)]. \tag{2.12}$$

Here we rewrite the derivative as $d/d\zeta=(n_1^2a^2\zeta/x)d/dx$, and take into account the symmetry properties $\int_{-\infty}^{\infty}dk\to 2\int_0^{\infty}dk$, $\int_{-\infty}^{\infty}d\zeta\to 2\int_0^{\infty}d\zeta$ to get

$$\mathcal{E}_m^{\mathrm{int}} = -\frac{n_1^2a^2}{2\pi^2}\int_0^{\infty}dk\int_0^{\infty}\frac{\zeta^2 d\zeta}{x}\times\frac{d}{dx}\ln[I_{mp}(x)I'_{mp}(x)]. \tag{2.13}$$

In the plane spanned by the axes $n_1\zeta$ and k we may introduce polar coordinates X and Y

$$X = n_1\zeta = \kappa_1 \cos\theta, \tag{2.14a}$$

$$Y = k = \kappa_1 \sin\theta, \tag{2.14b}$$

fulfilling the relation $X^2+Y^2=\kappa_1^2$. The area element in the XY plane becomes $\kappa_1 d\kappa_1 d\theta = n_1 d\zeta dk$. The integration of the polar angle over the first quadrant then becomes simple, $\int_0^{\pi/2}\cos^2\theta d\theta=\pi/4$, and we get

$$\begin{aligned}\mathcal{E}_m^{\mathrm{int}} &= -\frac{1}{8\pi n_1 a^2}\int_0^{\infty}dx x^2\frac{d}{dx}[I_{mp}(x)I'_{mp}(x)]\\ &= -\frac{1}{8\pi n_1 a^2}\int_0^{\infty}dx x^2\left[\frac{I'_{mp}(x)}{I_{mp}(x)} + \frac{I''_{mp}(x)}{I'_{mp}(x)}\right].\end{aligned} \tag{2.15}$$

Going back to Eq. (2.7) we can thus write the interior zero-point energy as

$$\begin{aligned}\mathcal{E}^{\mathrm{int}} = -\frac{1}{8\pi n_1 a^2}\Bigg\{&\sum_{m=1}^{\infty}\int_0^{\infty}x^2dx\left[\frac{I'_{mp}(x)}{I_{mp}(x)} + \frac{I''_{mp}(x)}{I'_{mp}(x)}\right]\\ &+ \int_0^{\infty}dx x^2\frac{I''_0(x)}{I'_0(x)}\Bigg\},\end{aligned} \tag{2.16}$$

where the last term represents the TE $m=0$ mode. No regularization procedure has been applied at this stage.

III. EXTERIOR REGION INCLUDED, ASSUMING PERFECTLY CONDUCTING CIRCULAR ARC

We now include the exterior region $r\geq a$, still assuming the circular arc at $r=a$ to be perfectly conducting.

A choice has to be made for what kind of medium to fill the space $r > a$. One possible choice might be to assume a vacuum on the outside. Another natural choice would be to take the exterior medium to be identical to the interior one. We will in this section allow for a generalization of the last option, namely, to assume that the exterior space is filled with a medium with arbitrary constants ϵ_2 and μ_2, but with the restriction that their product is the same as in the interior,

$$\epsilon_2\mu_2 = \epsilon_1\mu_1 = n^2. \tag{3.1}$$

We will refer to this situation as "diaphanous." This condition implying the constancy of light everywhere has under several occasions turned out to be convenient mathematically, for instance in connection with the Casimir theory for dielectric balls [31–37], and in the Casimir theory for the relativistic piecewise uniform string [38–42] (a review is given in Ref. [43]). In the latter case, the velocity of light is to be replaced with the velocity of sound. Condition (3.1) means in the present problem that λ takes the same value on the outside as on the inside (assuming k to take the same values on the two sides). The principal advantage of this assumption, which is not easily satisfied in nature, is that in simple cases Casimir self-energies will turn out then to be finite.

In the exterior region $r > a$ we have the expansions, keeping the formalism at first quite general,

$$E_r = \sum_{m=1}^{\infty}\left[-\frac{2k}{\lambda_2}H_{mp}^{(1)\prime}(\lambda_2 r)a_m^e - \frac{2i\mu_2\omega mp}{\lambda_2^2 r}H_{mp}^{(1)}(\lambda_2 r)b_m^e\right]\times F_0 \sin mp\theta, \tag{3.2a}$$

$$E_\theta = -\sum_{m=0}^{\infty}\left[\frac{2kmp}{\lambda_2^2 r}H_{mp}^{(1)}(\lambda_2 r)a_m^e + \frac{2i\mu_2\omega}{\lambda_2}H_{mp}^{(1)\prime}(\lambda_2 r)b_m^e\right]\times F_0 \cos mp\theta, \tag{3.2b}$$

$$E_z = 2i\sum_{m=1}^{\infty}H_{mp}^{(1)}(\lambda_2 r)a_m^e F_0 \sin mp\theta, \tag{3.2c}$$

$$H_r = \sum_{m=0}^{\infty}\left[\frac{2mp\epsilon_2\omega}{\lambda_2^2 r}H_{mp}^{(1)}(\lambda_2 r)a_m^e + \frac{2ik}{\lambda_2}H_{mp}^{(1)\prime}(\lambda_2 r)b_m^e\right]\times F_0 \cos mp\theta, \tag{3.2d}$$

$$H_\theta = -\sum_{m=1}^{\infty}\left[\frac{2\epsilon_2\omega}{\lambda_2}H_{mp}^{(1)\prime}(\lambda_2 r)a_m^e + \frac{2ikmp}{\lambda_2^2 r}H_{mp}^{(1)}(\lambda_2 r)b_m^e\right]\times F_0 \sin mp\theta, \tag{3.2e}$$

$$H_z = 2\sum_{m=0}^{\infty}H_{mp}^{(1)}(\lambda_2 r)b_m^e F_0 \cos mp\theta. \tag{3.2f}$$

As before, F_0 is given by Eq. (2.3). The presence of the Hankel function of the first kind, $H_{mp}^{(1)}$, ensures proper behavior (outgoing waves) at infinity. The e superscript refers to exterior modes.

Let us now take into account condition (3.1), implying $\lambda_1 = \lambda_2 \equiv \lambda$, and consider the boundary conditions. For the TM polarization (the a_m^e modes) we get

$$H_{mp}^{(1)}(\lambda a) = 0, \quad m \ge 1, \tag{3.3}$$

whereas for the TE polarization (the b_m^e modes),

$$H_{mp}^{(1)\prime}(\lambda a) = 0, \quad m \ge 0. \tag{3.4}$$

The roots of these eigenvalue equations are complex—nevertheless, the argument principle may be applied as has been explained in detail in many places [17,44,45]. We can now calculate the exterior zero-point energy $\mathcal{E}^{\text{ext}}$ in the same way as above. The modified Bessel function K_ν is introduced via $H_\nu^{(1)}(ix) = (2/\pi)i^{-(\nu+1)}K_\nu(x)$. For the total zero-point energy/length $\mathcal{E} = \mathcal{E}^{\text{int}} + \mathcal{E}^{\text{ext}}$ we obtain

$$\mathcal{E} = -\frac{1}{8\pi n a^2}\left\{\sum_{m=1}^{\infty}\int_0^\infty x^2 dx\left[\frac{I_{mp}'(x)}{I_{mp}(x)} + \frac{I_{mp}''(x)}{I_{mp}'(x)} + \frac{K_{mp}'(x)}{K_{mp}(x)} + \frac{K_{mp}''(x)}{K_{mp}'(x)}\right] + \int_0^\infty x^2 dx\left[\frac{I_0''(x)}{I_0'(x)} + \frac{K_0''(x)}{K_0'(x)}\right]\right\}. \tag{3.5}$$

We now must face up to the fact that our result contains an irremovable divergence, associated with the nonzero a_2 heat kernel coefficient found by Nesterenko *et al.* [8,9]. This occurs precisely because of the $m=0$ terms in Eq. (3.5). If we were to write that expression as

$$\mathcal{E} = -\frac{1}{8\pi n a^2}\left\{{\sum_{m=0}^{\infty}}'\int_0^\infty x^2 dx\left[\frac{I_{mp}'(x)}{I_{mp}(x)} + \frac{I_{mp}''(x)}{I_{mp}'(x)} + \frac{K_{mp}'(x)}{K_{mp}(x)} + \frac{K_{mp}''(x)}{K_{mp}'(x)}\right] - \frac{1}{2}\int_0^\infty x^2 dx \frac{d}{dx}\ln\left(\frac{I_0(x)}{I_0'(x)}\frac{K_0(x)}{K_0'(x)}\right)\right\} = \tilde{\mathcal{E}} + \hat{\mathcal{E}}, \tag{3.6}$$

where the prime on the summation sign means that the $m=0$ terms are counted with half weight, we see in the following that the summation, $\tilde{\mathcal{E}}$, may now be rendered finite (see Appendix A), but the residual correction, $\hat{\mathcal{E}}$, is divergent.

It is instructive to break up this residual zero-mode contribution into its Dirichlet (TM) and Neumann (TE) parts. The former involves, asymptotically for large x

$$I_0(x)K_0(x) \sim \frac{1}{2x}\left[1 + \frac{1}{8x^2} + O\left(\frac{1}{x^4}\right)\right], \tag{3.7a}$$

while the latter requires

$$I_0'(x)K_0'(x) \sim -\frac{1}{2x}\left[1 - \frac{3}{8x^2} + O\left(\frac{1}{x^4}\right)\right]. \tag{3.7b}$$

Then the two contributions to the residual zero-mode terms are

$$-\frac{1}{2}\int_0^\infty dx x^{2-s}\frac{d}{dx}\ln[I_0(x)K_0(x)] \sim \frac{1}{2}\int_0^\infty dx x^{2-s}\left(\frac{1}{x} + \frac{1}{4x^3} + \dots\right), \tag{3.8a}$$

$$\frac{1}{2}\int_0^\infty dx x^{2-s}\frac{d}{dx}\ln[I_0'(x)K_0'(x)] \sim \frac{1}{2}\int_0^\infty dx x^{2-s}\left(-\frac{1}{x}+\frac{3}{4x^3}+\ldots\right), \tag{3.8b}$$

so the $1/x$ terms cancel between the two modes (alternatively those terms may be removed by contact terms, as we will see in the following), but the subleading $1/x^3$ terms constitute an irremovable logarithmic divergence. Here, we have indicated an analytic regularization by taking s to zero through positive values, which corresponds to the following divergent terms as $s\to 0$:

$$-\frac{1}{2}\int_0^\infty dx x^{2-s}\frac{d}{dx}\ln I_0(x)K_0(x) \sim \frac{1}{8s}, \tag{3.9a}$$

$$\frac{1}{2}\int_0^\infty dx x^{2-s}\frac{d}{dx}\ln I_0'(x)K_0'(x) \sim \frac{3}{8s}. \tag{3.9b}$$

These precisely correspond to the two mode contributions, adding up to $1/2s$, found by Nesterenko *et al.* [8].

This zero-mode divergence is due to the sharp corners where the arc meets the wedge. We will proceed by setting this term aside, and computing the balance of the Casimir free energy. We note there is a closely related problem which Nesterenko *et al.* [9] dubbed a cone. That is, we identify the two wedge boundaries at $\theta=0$ and α, and impose periodic boundary conditions there. This means that we may take the angular function in the mode sums to be $e^{imp\theta}$, where m may be either positive or negative, and where now $p=2\pi/\alpha$. Now all modes, including the zero modes ($m=0$) contribute equally, and the summation on m becomes

$$\sum_{m=-\infty}^{\infty} = 2\sum_{m=0}^{\infty}{}' \tag{3.10}$$

with the zero modes both having 1/2 weight in the latter form. (For the radial function in the interior we can only use $I_{|\nu|}$ in order that the solution be finite at the origin.) Thus we get precisely $2\tilde{\mathcal{E}}$ [Eq. (3.6)] without the residual zero-mode term $\hat{\mathcal{E}}$, and we have eliminated the irremovable logarithmic divergence. This is because the sharp corners, where the arc meets the wedge, have been removed because there is no wedge boundary. So if the reader prefers, he or she may regard the rest of the discussion in this and the following section to refer to this situation, which will introduce an additional factor of two into the Casimir free energy, and with the restriction $p\geq 1$, where $p=1$ corresponds to the circular cylinder first considered in Ref. [17].

So in any case disregarding in the following the residual zero-mode pieces $\hat{\mathcal{E}}$, we consider now the regularization of the $\Sigma_{m=0}^{\infty\prime}$ terms in Eq. (3.6), $\tilde{\mathcal{E}}$, which, in order to be a Casimir energy, ought to be given in such a form that it reduces to zero in the limit when $a\to\infty$. This will eliminate the divergence associated with the apex, which is not relevant to the force on the circular arc. It is easy to satisfy this requirement by observing that for large values of x, and for general ν, we can approximate

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}}e^x, \quad K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}, \quad x\to\infty, \tag{3.11}$$

implying that $I_\nu'\sim I_\nu$ and $K_\nu'\sim -K_\nu$. Accordingly,

$$\frac{d}{dx}\ln(-I_\nu K_\nu I_\nu' K_\nu') \sim 2\frac{d}{dx}\ln(I_\nu K_\nu) \sim -\frac{2}{x} \tag{3.12}$$

to leading order in x. This term is to be subtracted off from the integrand in Eq. (3.5). The Casimir energy for the wedge becomes then

$$\tilde{\mathcal{E}} = -\frac{1}{8\pi n a^2}\sum_{m=0}^{\infty}{}'\int_0^\infty x^2 dx\left[\frac{I_{mp}'(x)}{I_{mp}(x)}+\frac{I_{mp}''(x)}{I_{mp}'(x)}+\frac{K_{mp}'(x)}{K_{mp}(x)} + \frac{K_{mp}''(x)}{K_{mp}'(x)}+\frac{2}{x}\right]. \tag{3.13}$$

We may here perform a partial integration (the boundary terms at $x=0$ and $x=\infty$ do not contribute),

$$\tilde{\mathcal{E}} = \frac{1}{4\pi n a^2}\sum_{m=0}^{\infty}{}'\int_0^\infty x dx \times \ln[-4x^2 I_{mp}(x)I_{mp}'(x)K_{mp}(x)K_{mp}'(x)]. \tag{3.14}$$

It is helpful to introduce a quantity $\lambda_\nu(x)$ for arbitrary order ν,

$$\lambda_\nu(x) = [I_\nu(x)K_\nu(x)]', \tag{3.15}$$

and to take into account the Wronskian $W\{I_\nu, K_\nu\}=-1/x$. From this we calculate the following useful relationship:

$$-4x^2 I_\nu(x)I_\nu'(x)K_\nu(x)K_\nu'(x) = 1-x^2\lambda_\nu^2(x), \tag{3.16}$$

and so end up with the following convenient form for the Casimir energy,

$$\tilde{\mathcal{E}} = \frac{1}{4\pi n a^2}\sum_{m=0}^{\infty}{}'\int_0^\infty x dx \ln[1-x^2\lambda_{mp}^2(x)]. \tag{3.17}$$

This is thus the boundary-induced contribution to the zero-point energy. If the boundary $r=a$ were removed and either the interior or the exterior medium were chosen to fill the whole wedge region, we would get $\tilde{\mathcal{E}}=0$. This is a property relying on condition (3.1) above. The temperature is assumed to be zero.

Although the leading behavior of the Bessel functions has been subtracted in Eq. (3.17), it is still not in general finite. We will see in the following section how a finite self-energy may be extracted from this formula. For now, we observe that this is a generalization of the standard formal result for a conducting circular cylinder, which is obtained from this result in the special case $p=1$ [17]. (The overall $1/n$ comes from an elementary scaling argument [46].) When $p=1$ expression (3.17) is one-half that for a conducting circular cylinder. Referring to the perfectly conducting wedge boundaries, we see that the Casimir energy for periodic boundary conditions, with period 2π, is twice the Casimir energy for a perfectly conducting boundary condition imposed on the θ interval of π, a result obvious from the replacement of $e^{im\theta}$ for m of either sign in the former case by $\sin m\theta$ or $\cos m\theta$, $m\geq 0$, in the latter. This general observation, which is the theorem stated in Eq. (2.49) of Ref. [47] (see also Ref. [48]) will allow us to obtain numerical results rather immediately. For the periodic boundary-condition situation, which eliminates the zero-mode problem, $p=1$ is exactly the circular cylinder problem, and there is no additional factor of 1/2.

IV. DIELECTRIC BOUNDARY AT $r=a$

Assume now that the perfectly conducting arc at $r=a$ is removed and replaced by a dielectric boundary, wherewith the interior and exterior regions become coupled via electromagnetic boundary conditions at $r=a$. As before, we assume that the plane surfaces $\theta=0$ and $\theta=\alpha$ are perfectly conducting for all values of r. (Alternatively, we may impose periodic boundary conditions there.)

We shall assume in the following that the media are arbitrary, with real and constant parameters ϵ_1,μ_1 in the interior and ϵ_2,μ_2 in the exterior, without any restriction imposed on their product. This will, however, result in general in a divergent Casimir self-energy.

Let λ_2 be the transverse wave number in the exterior region,

$$\lambda_2^2 = n_2^2\omega^2 - k^2, \tag{4.1}$$

with $n_2^2=\epsilon_2\mu_2$. The basic expansions are Eqs. (2.1a), (2.1b), (2.1c), (2.1d), (2.1e), and (2.1f) in the interior and Eqs. (3.2a), (3.2b), (3.2c), (3.2d), (3.2e), and (3.2f) in the exterior.

As for the boundary conditions at $r=a$, only the tangential field components have to be taken into account. From the continuity of E_z and H_z we get, respectively,

$$J_{mp}(u)a_m^i = H_{mp}^{(1)}(v)a_m^e \tag{4.2}$$

and

$$J_{mp}(u)b_m^i = H_{mp}^{(1)}(v)b_m^e, \tag{4.3}$$

where we have defined

$$u=\lambda_1 a, \quad v=\lambda_2 a. \tag{4.4}$$

From the component E_θ we get

$$\frac{kmp}{u^2}J_{mp}(u)a_m^i + \frac{i\mu_1\omega}{u}J'_{mp}(u)b_m^i = \frac{kmp}{v^2}H_{mp}^{(1)}(v)a_m^e + \frac{i\mu_2\omega}{v}H_{mp}^{(1)\prime}(v)b_m^e, \tag{4.5}$$

and from the component H_θ,

$$\frac{i\epsilon_1\omega}{u}J'_{mp}(u)a_m^i - \frac{kmp}{u^2}J_{mp}(u)b_m^i = \frac{i\epsilon_2\omega}{v}H_{mp}^{(1)\prime}(v)a_m^e - \frac{kmp}{v^2}H_{mp}^{(1)}(v)b_m^e. \tag{4.6}$$

The two last equations mean that a superposition of the TM and TE waves is in general necessary to satisfy the boundary conditions. The exception is the axially symmetric case $m=0$. The condition for solution of the set of linear equations is that the system determinant vanishes. Observing the relation

$$u^2 - v^2 = (n_1^2 - n_2^2)\omega^2 a^2 \tag{4.7}$$

which follows from Eqs. (4.1) and (4.4), we obtain after some manipulations the condition

$$\left[\frac{\mu_1}{u}\frac{J'_{mp}(u)}{J_{mp}(u)} - \frac{\mu_2}{v}\frac{H_{mp}^{(1)\prime}(v)}{H_{mp}^{(1)}(v)}\right]\left[\frac{\epsilon_1\omega^2}{u}\frac{J'_{mp}(u)}{J_{mp}(u)} - \frac{\epsilon_2\omega^2}{v}\frac{H_{mp}^{(1)\prime}(v)}{H_{mp}^{(1)}(v)}\right] = m^2p^2k^2\left(\frac{1}{v^2}-\frac{1}{u^2}\right)^2. \tag{4.8}$$

This is essentially the same transcendental eigenvalue equation as found for a step-index optical fiber (cf., for instance, Ref. [30] or [49]). In transmission problems, one is usually interested in calculating the discrete values of the propagation constant k, assuming that the waveguide is fed with some frequency ω. Here our intention is different, namely, to calculate the discrete values of ω on the basis of an input value for the continuous axial wave vector k. As we noted in Sec. III, this dispersion relation generalizes that for a circular cylinder, the special case $p=1$.

It may be noted that the roots of Eq. (4.8) are both real and complex. Application of the argument principle to such a problem is discussed in Ref. [50].

A. $n_1=n_2$

The TE and TM modes decouple in the special case when $n_1=\sqrt{\epsilon_1\mu_1}=n_2=\sqrt{\epsilon_2\mu_2}$. In this case, dispersion relation (4.8) reduces to $\Delta\tilde{\Delta}=0$, where Δ and $\tilde{\Delta}$ are the two factors on the left-hand side of Eq. (4.8), and then using the Wronskian, we find after Euclidean rotation, $\omega\to i\zeta$,

$$\Delta\tilde{\Delta} = \frac{(\epsilon_1+\epsilon_2)}{4c^2\epsilon_1\epsilon_2}\frac{1-\xi^2x^2(I_{mp}K_{mp})'^2}{x^2I_{mp}^2(x)K_{mp}^2(x)}, \tag{4.9}$$

where $x=\kappa a$, $c=1/n$, and the reflection coefficient (for either polarization)

$$\xi = \frac{\epsilon_2-\epsilon_1}{\epsilon_2+\epsilon_1}. \tag{4.10}$$

We conclude that the formula for the (zero-mode subtracted) Casimir energy is

$$\tilde{\mathcal{E}} = \frac{1}{2}\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dk}{2\pi}{\sum_{m=0}^{\infty}}'\int_{i\infty}^{-i\infty}d\omega\,\omega\frac{d}{d\omega}\ln g_m(x), \tag{4.11}$$

where

$$g_m(x) = 1 - \xi^2 x^2 \lambda_{mp}^2, \tag{4.12}$$

where λ_{mp} is given by Eq. (3.15). Here, we have again subtracted off the terms that would be present if either medium filled the entire wedge. [The divergence structure of the zero-mode term subtracted from Eq. (4.11) is analyzed in Appendix B.] Again cavalierly integrating by parts, we obtain, using the change of variables (2.14),

$$\tilde{\mathcal{E}} = \frac{1}{4\pi n a^2} \sum_{m=0}^{\infty}{}' \int_0^\infty dx x \ln[1 - \xi^2 x^2 \lambda_{mp}^2]. \tag{4.13}$$

As expected, this differs from conducting case (3.17) by the appearance of ξ^2 in front of λ_{mp}. The conducting case is obtained by setting $\xi=1$. All of this is just as for the circular cylinder case, which is obtained from the $p=1$ result by multiplying by a factor of 2.

Let us now extract both the $\xi=1$ (perfect conducting) and the small ξ results for arbitrary p. A simple route is to follow the method given in [18] or in Chap. 7 of Ref. [47]. The point is simply that the uniform asymptotic expansion for the modified Bessel functions yields an asymptotic expansion for large p. Thus we can write (see the Appendix A for details)

$$2n\tilde{\mathcal{E}} = \frac{\xi^2}{16\pi a^2} \ln(2\pi/p) + \bar{\mathcal{E}}_0 + 2\sum_{m=1}^{\infty} \bar{\mathcal{E}}_m, \tag{4.14}$$

where

$$\bar{\mathcal{E}}_0 = \frac{1}{4\pi a^2} \int_0^\infty dx x \left[\ln(1 - \xi^2 x^2 \lambda_0^2(x)) + \frac{\xi^2}{4} \frac{x^4}{(1+x^2)^3} \right], \tag{4.15a}$$

$$\bar{\mathcal{E}}_m = \frac{1}{4\pi a^2} \int_0^\infty dx x \left[\ln(1 - \xi^2 x^2 \lambda_{mp}^2(x)) + \frac{\xi^2}{4} \frac{x^4}{(m^2 p^2 + x^2)^3} \right]. \tag{4.15b}$$

(Further details are given in the cited references.) Because of the subtractions in the integrals, they are convergent.

Let us first consider ξ as small, and keep only the terms of order ξ^2. Using the uniform asymptotic approximants, we find for large mp,

$$\bar{\mathcal{E}}_m \sim \frac{\xi^2}{4\pi a^2} \left(\frac{1}{96 m^2 p^2} - \frac{7}{3840 m^4 p^4} + \cdots \right), \tag{4.16}$$

while numerical integration gives

$$\bar{\mathcal{E}}_0 = \frac{\xi^2}{4\pi a^2} (-0.490\,877\,5). \tag{4.17}$$

Thus

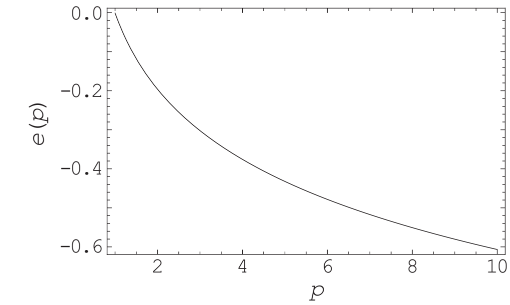

FIG. 3. Casimir energy for weak coupling, $\xi^2 \ll 1$, as a function of p which is related to the dihedral angle $\alpha = \pi/p$. This graph shows the energy for $p > 1$.

$$\tilde{\mathcal{E}} = \frac{\xi^2}{8\pi n a^2} \left(-0.490\,877\,5 + \frac{1}{4} \ln 2\pi/p + \frac{\pi^2}{288} \frac{1}{p^2} - \frac{7\pi^4}{172800} \frac{1}{p^4} + 2\sum_{1}^{M} [f(mp) - g(mp)] \right) \equiv \frac{\xi^2}{8\pi n a^2} e(p), \tag{4.18}$$

where we have added and subtracted the first two terms in the uniform asymptotic expansion,

$$g(\nu) = \frac{1}{96\nu^2} - \frac{7}{3840\nu^4}, \tag{4.19}$$

and f is the integral appearing in $\bar{\mathcal{E}}_m$,

$$f(\nu) = \int_0^\infty dx x^3 \left[-\lambda_\nu^2 + \frac{1}{4} \frac{x^2}{(x^2+\nu^2)^3} \right]. \tag{4.20}$$

In principle we are to take the $M \to \infty$ limit in Eq. (4.18). In practice, we may keep only a few terms in the m sum. For example, keeping none of those corrections, that is setting $M=0$, we get for $p=1$, $e(1) \approx -0.001\,084\,7$. Keeping three terms is sufficient to find that $e(1)$ is less than 1×10^{-6}; indeed, the circular cylinder value is $e(1)=0$ [21–25]. This function $e(p)$ is plotted in Fig. 3, for $p>1$, where it is sufficient to keep the leading asymptotic approximations; for p between 1/2 and 1 (α between π and 2π) we must retain at least one correction, $M=1$, as shown in Fig. 4. (No observable change occurs with larger M.) Numerically, we see that

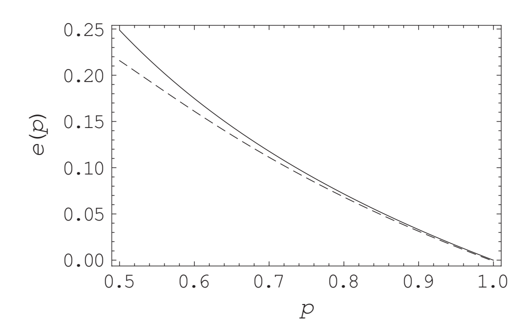

FIG. 4. Casimir energy for weak coupling, $\xi^2 \ll 1$, as a function of p which is related to the dihedral angle $\alpha = \pi/p$. This graph shows the energy for $0.5 < p < 1$. The upper curve shows the exact energy, the lower the leading asymptotic approximation, obtained from Eq. (4.18) by setting $M=0$.

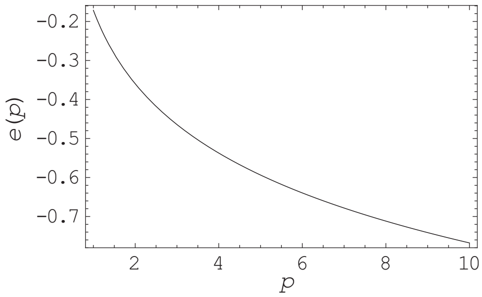

FIG. 5. Casimir energy for strong coupling, $\xi^2=1$, as a function of p, related to the dihedral angle by $\alpha=\pi/p$, for $p>1$. In this region $M=0$ in Eq. (4.22) is sufficient.

the value for a cylinder with a conducting septum ($p=1/2$) is indistinguishable from $e(0.5)=1/4$.

Recall for periodic boundary conditions on the wedge (the "cone") $p=2\pi/\alpha\geq 1$, and an additional factor of two appears in the energy. Similarly, following the same references, we can obtain the strong coupling (perfect conductor) limit, $\xi=1$. This time the formula for the energy is

$$\tilde{\mathcal{E}}=\frac{1}{8\pi n a^2}e(p), \tag{4.21}$$

where

$$e(p)=-0.651\,752+\frac{1}{4}\ln 2\pi/p+\frac{7\pi^2}{2880}\frac{1}{p^2}-\frac{\pi^4}{32256}\frac{1}{p^4}+2\sum_{m=1}^{M}[f(mp)-g(mp)], \tag{4.22}$$

where again the limit $M\to\infty$ is understood. Now f is given by

$$f(\nu)=\int_0^\infty dx x\left[\ln(1-x^2\lambda_\nu^2)+\frac{1}{4}\frac{x^4}{(x^2+\nu^2)^3}\right], \tag{4.23}$$

and now the asymptotic terms are

$$g(\nu)=\frac{7}{960\nu^2}-\frac{5}{3584\nu^4}. \tag{4.24}$$

Keeping no correction terms is already very good at $p=1$, where with $M=0$ $e(1)/(4\pi)\approx -0.013\,633$, only slightly different from the exact answer of $-0.013\,56$ [17]. Keeping just $M=1$ gives exact coincidence to the indicated accuracy. This function $e(p)$ is plotted in Fig. 5 for $p>0$ where the asymptotic approximation is sufficient, while two correction terms are included in the region $0.5<p<1$, as shown in Fig. 6. It is curious that the energy vanishes now not at $p=1$, but at $p=0.583$.

(Again, recall only $p\geq 1$ is relevant for periodic boundary conditions on the wedge.)

B. $n_1\neq n_2$, $\mu_1=\mu_2=1$

Finally, we can follow Ref. [21] to obtain the weak-coupling Casimir self-energy for a purely dielectric wedge, where $\mu_1=\mu_2=1$. We can only examine the coefficient of $(\epsilon_1-\epsilon_2)^2$ because the result is divergent in higher orders. It is hardly necessary to give details since all that is necessary is to replace m by mp in the analysis given in that reference. The energy per area in the wedge is

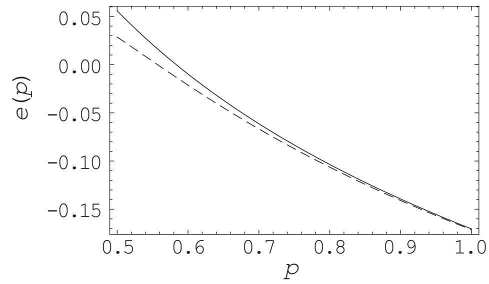

FIG. 6. Casimir energy for strong coupling, $\xi^2=1$, as a function of p, related to the dihedral angle by $\alpha=\pi/p$, for $0.5<p<1$. In this region $M=2$ in Eq. (4.22) is sufficient, and the comparison with the $M=0$ result (lower curve) is made.

$$\tilde{\mathcal{E}}=\frac{(\epsilon_1-\epsilon_2)^2}{32\pi n a^2}\sum_{m=0}^{\infty}{}' \int_0^\infty dy y^4 g_m(y), \tag{4.25}$$

where the exact form of g_m is elaborate, but has the asymptotic form

$$g_m(y)\sim\frac{1}{2m^2p^2}\sum_{k=1}^{\infty}\frac{1}{(mp)^k}f_k(z),\quad mp\to\infty, \tag{4.26}$$

where $y=mpz$, and f_k are rational functions of z, given in Ref. [21], about which all we need to know here is

$$p^2\lim_{s\to 0}\sum_{m=1}^{\infty}m^{2-s}\int_0^\infty dz z^{4-s}f_1(z)=-p^2\frac{\zeta(3)}{16\pi^2}, \tag{4.27a}$$

$$\int_0^\infty dz\left[z^4f_2(z)-\frac{1}{8}\right]=0, \tag{4.27b}$$

$$\lim_{s\to 0}\sum_{m=0}^{\infty}{}'(mp)^{-s}\int_0^\infty dz z^{4-s}f_3(z)=\frac{5}{32}\ln 2\pi/p, \tag{4.27c}$$

$$\int_0^\infty dz z^4 f_4(z)=0, \tag{4.27d}$$

$$\frac{1}{p^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\int_0^\infty dz z^4 f_5(z)=\frac{19\pi^2}{7680}\frac{1}{p^2}, \tag{4.27e}$$

$$\int_0^\infty dz z^4 f_6(x)=0, \tag{4.27f}$$

$$\frac{1}{p^4}\sum_{m=1}^{\infty}\frac{1}{m^4}\int_0^\infty dz z^4 f_7(z)=-\frac{209\pi^4}{5\,806\,080}\frac{1}{p^4}. \tag{4.27g}$$

Here a contact term, which cannot contribute to any observable force, has been removed from Eq. (4.27b). Again, for

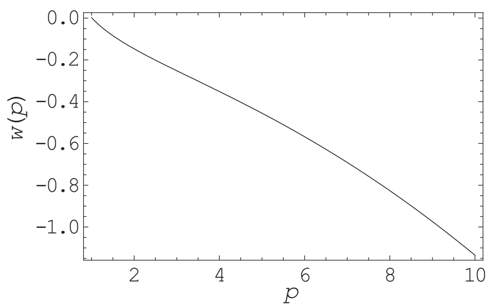

FIG. 7. Casimir energy for weak coupling, $|\epsilon_1-\epsilon_2|\ll 1$, as a function of p, which is related to the dihedral angle $\alpha=\pi/p$, for a purely dielectric wedge. Here, for $p>1$, only the leading asymptotic terms have been included.

the precise definition of Eq. (4.27c) see Appendix A. Then the Casimir energy per unit length of the dilute dielectric wedge is

$$\widetilde{\mathcal{E}} \sim \frac{(\epsilon_1-\epsilon_2)^2}{64\pi n a^2} w(p), \tag{4.28}$$

where

$$w(p) \approx -p^2\frac{\zeta(3)}{16\pi^2} + \frac{5}{32}\ln(2\pi/p) + \frac{19\pi^2}{7680p^2} - 0.301590 + \sum_{m=1}^{4} r(mp) - \frac{0.000\,012}{p^2}, \tag{4.29}$$

where

$$r(\nu) = 2\int_0^\infty dy y^4\left[g_\nu(y) - \frac{1}{2\nu^2}\sum_{k=1}^{5}\frac{1}{\nu^k}f_k(y/\nu)\right], \tag{4.30}$$

and we have used the next term in the asymptotic series to estimate the contribution for $m\geq 5$. This, numerically, yields the correct value of zero for $p=1$. The values for the Casimir energy for larger values of p are shown in Fig. 7, and for smaller values of p in Fig. 8.

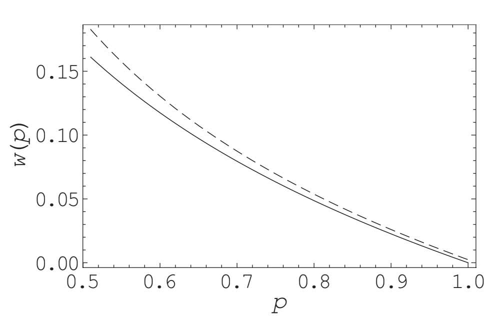

FIG. 8. Casimir energy for weak coupling, $|\epsilon_1-\epsilon_2|\ll 1$, as a function of p, which is related to the dihedral angle $\alpha=\pi/p$, for a purely dielectric wedge. Here, for $0.5<p<1$, the effect of the remainder is significant; the upper curve shows only the leading asymptotic terms, while the lower curve includes the remainder function r.

For the septum case, the numerical value of $w(0.5)=0.1666$, which seems likely to represent exactly 1/6. (Periodic boundary conditions restrict $p\geq 1$.)

V. ENERGY PRODUCTION IN THE SUDDEN FORMATION OF A COSMIC STRING

As already mentioned, the electromagnetic theory of the wedge is related to the theory of cosmic strings. In general, cosmic strings are believed to be possible ingredients in the very early Universe; they are related to phase transitions. One particular aspect of this study is to estimate the energy production in the form of massless particles when a string is formed "suddenly" at some instant $t=t_0$, where t_0 is a characteristic time usually taken to be of order 1×10^{-40} s as is typical for grand unified theories (GUTs). One can calculate the number of particles associated with the formation of the string in terms of Bogoliubov coefficients relating the initial Minkowski metric to the (static) metric after the string has been formed. This approach was pioneered by Parker [51] for massless scalar fields, assuming the string radius to be zero. Analogous calculations were made in Ref. [52] in the electromagnetic case, still assuming the string radius to be zero, and in Ref. [53] for the case of a finite string radius. (See also Ref. [28].)

In the following we shall investigate the following model: Let a cosmic string of vanishing radius and large length L be formed suddenly along the cusp line of the wedge, i.e., along the z axis. The time of formation is $t=t_0$. In the interior region we assume that there is one single isotropic medium present, with refractive index $n=\sqrt{\epsilon\mu}$. The interior region is closed by a perfectly conducting arc with radius $r=a$. In the present context, a plays the role of a "large" boundary; in connection with strings, a is usually taken to be of the same order as L [51]. The finiteness of a will moreover make it possible to normalize the fundamental modes in a conventional way. As for the description of the electromagnetic fields, we have to distinguish between the Minkowski metric present for $t<t_0$ and the string metric for $t>t_0$.

A. $t<t_0$ case

We shall consider the TM mode only, for which the central field component is E_z. For definiteness we reproduce the fields in the m,k mode here, in a convenient notation (replacing the previous combination $2ia_m^i$ with the symbol N)

$$E_z = NJ_{mp}(\lambda r)F_0 \sin mp\theta, \tag{5.1}$$

$$E_r = \frac{ik}{\lambda}NJ'_{mp}(\lambda r)F_0 \sin mp\theta, \tag{5.2}$$

$$E_\theta = \frac{ikmp}{\lambda^2 r}NJ_{mp}(\lambda r)F_0 \cos mp\theta, \tag{5.3}$$

$$H_z = 0, \tag{5.4}$$

$$H_r = -\frac{imp\epsilon\omega}{\lambda^2 r}NJ_{mp}(\lambda r)F_0 \cos mp\theta, \tag{5.5}$$

$$H_\theta = \frac{i\omega\epsilon}{\lambda} N J'_{mp}(\lambda r) F_0 \sin mp\theta. \tag{5.6}$$

As before, $\lambda=\sqrt{n^2\omega^2-k^2}$, $F_0=\exp(ikz-i\omega t)$, and $m\geq 1$. The boundary condition on the arc is $J_{mp}(\lambda a)=0$, giving the solutions λ_{ms}, $s=1,2,3,\ldots$ for the transverse wave number λ.

It is now convenient as an intermediate step to make use of the formalism of scalar field theory. Define the scalar field mode ψ_{msk}, satisfying Dirichlet boundary conditions on all surfaces, as

$$\psi_{msk} = N J_{mp}(\lambda_{ms} r) F_0 \sin mp\theta. \tag{5.7}$$

It is seen to have the same form as the m,s,k mode of the field component E_z. For reasons to become clear later, we choose the magnitude $|N|$ of the normalization constant N to be

$$|N| = \frac{1}{n}\sqrt{\frac{2}{\alpha\epsilon\omega_{msk}}}\frac{\lambda_{ms}}{a|J_{mp+1}(\lambda_{ms}a)|}, \tag{5.8}$$

with $\omega_{msk}=(1/n)\sqrt{\lambda_{ms}^2+k^2}$.

We define the Klein-Gordon product as

$$(\psi_{msk},\psi_{m's'k'}) = \frac{-i\epsilon n^2}{\lambda_{ms}^2}\int \psi_{msk}\overleftrightarrow{\partial}_0\psi^*_{m's'k'} r dr d\theta dz, \tag{5.9}$$

and then get by direct calculation

$$(\psi_{msk},\psi_{m's'k'}) = 2\pi\delta(k-k')\delta_{mm'}\delta_{ss'}. \tag{5.10}$$

Consider now the electromagnetic energy W in the wedge region. We may calculate it by integrating the energy density w over the volume,

$$W = \int w dV = \frac{1}{4}\int [\epsilon|E|^2 + \mu|H|^2] r dr d\theta dz, \tag{5.11}$$

using the general recursion equation $[J_\nu = J_\nu(x)]$

$$(J'_\nu)^2 + \frac{\nu^2}{x^2}J_\nu^2 = \frac{1}{2}(J_{\nu-1}^2 + J_{\nu+1}^2), \tag{5.12}$$

as well as the integral formula

$$\int_0^a [J_{mp-1}^2(\lambda_{ms}r) + J_{mp+1}^2(\lambda_{ms}r)] r dr = a^2 J_{mp+1}^2(\lambda_{ms}a), \tag{5.13}$$

which holds when $J_{mp}(\lambda_{ms}a)=0$. It is however simpler to go via the axial energy flux P, given as

$$P = \int S_z dA, \tag{5.14}$$

where $dA = r dr d\theta$ is the cross-sectional area element, and where

$$S_z = \frac{1}{2}\Re(E_r H_\theta^* - E_\theta H_r^*) \tag{5.15}$$

is the Poynting vector. As in any linear wave theory we can set [54]

$$P = \frac{W}{L}c_g, \tag{5.16}$$

where c_g is the axial group velocity. From Eqs. (5.14) and (5.15) we then get

$$P = \frac{\alpha\epsilon k a^2\omega_{mks}}{8\lambda_{ms}^2}|N|^2 J_{mp+1}^2(\lambda_{ms}a). \tag{5.17}$$

In geometric units P has the dimension cm^{-2}. As $c_g = d\omega/dk = k/(n^2\omega)$ we get for the energy per unit length

$$\frac{W}{L} = \frac{\alpha\epsilon n^2 a^2\omega_{msk}^2}{8\lambda_{ms}^2}|N|^2 J_{mp+1}^2(\lambda_{ms}a). \tag{5.18}$$

We see that W/L is expressible in terms of $|E_z|^2$ integrated over the cross section,

$$\frac{W}{L} = \frac{\epsilon n^2\omega_{msk}^2}{2\lambda_{ms}^2}\int |E_z|^2 dA. \tag{5.19}$$

This relation will turn out to be useful in the following.

Quantum theory. We assume henceforth the real representation for the fields. The component $E_z(\mathbf{r},t)\equiv E_z(x)$, considered quantum mechanically as a Hermitian operator, is expanded as

$$E_z(x) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{m=1}^{\infty}\sum_s [a_{msk}\psi_{msk}(x) + a_{msk}^\dagger\psi_{msk}^*(x)], \tag{5.20}$$

where a_{msk} and $a_{msk}^\dagger$ are annihilation and creation operators satisfying the commutation relations

$$[a_{msk}, a_{m's'k'}^\dagger] = 2\pi\delta(k-k')\delta_{mm'}\delta_{ss'}. \tag{5.21}$$

We now go back to relation (5.19), and require that the total energy W associated with the m,k,s mode is equal to the occupation number $\langle a_{msk}^\dagger a_{msk}\rangle$ times the photon energy ω_{msk},

$$\frac{\epsilon n^2\omega_{msk}^2}{\lambda_{ms}^2}\int \langle E_z^2\rangle r dr d\theta dz = \left\langle a_{msk}^\dagger a_{msk} + \frac{1}{2}\right\rangle \omega_{msk}. \tag{5.22}$$

Here we insert expansion (5.20). Because of the orthogonality of Eq. (5.21), the various modes decouple so that the total energy is a sum over the mode energies. For the mode ψ_{msk}, written in the form of Eq. (5.7), we get from the condition above the expression for the normalization constant $|N|$ already given in Eq. (5.8). If $n=1$ and $\alpha=2\pi$, we recover the expression given in Ref. [52].

B. $t>t_0$ case: The Bogoliubov transformation

After the sudden creation of the cosmic string along the cusp line (the z axis) at the instant $t=t_0$, we assume that the string metric is static. All transient phenomena are intended to be taken care of via the use of the quantum mechanical sudden transformation below. We first have to establish the field expressions in the presence of the string metric. The central gravitational quantity appearing in the formalism will be

$$\beta = (1-4GM)^{-1}, \tag{5.23}$$

already introduced above in Eq. (1.2b). In a string context, β is believed to be very close to unity. Writing the field component E_z as $E_z(r)\exp(ikz-i\omega t)\sin mp\theta$ we obtain the following equation for the quantity $E_z(r)$:

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}+\lambda^2-\frac{\beta^2m^2p^2}{r^2}\right)E_z(r)=0, \tag{5.24}$$

with $\lambda^2=n^2\omega^2-k^2$ as before. Introducing the symbol ν as

$$\nu=\beta m, \tag{5.25}$$

we can write the fundamental ν,s,k mode as

$$\psi_{\nu sk}=N_\nu J_{\nu p}(\lambda_{\nu s}r)F_0 \sin mp\theta, \tag{5.26}$$

with $F_0=\exp(ikz-i\omega_{\nu sk}t)$. The boundary condition on $r=a$ is $J_{\nu p}(\lambda a)=0$, giving solutions $\lambda_{\nu s}$, $s=1,2,3\ldots$ for the transverse wave number.

The formalism now becomes quite similar to that given before in the nongravitational case. We list the main formulas. The normalization constant $|N_\nu|$ becomes

$$|N_\nu|=\frac{1}{n}\sqrt{\frac{2\beta}{\alpha\epsilon\omega_{\nu sk}}}\frac{\lambda_{\nu s}}{a|J_{\nu p+1}(\lambda_{\nu s}a)|}, \tag{5.27}$$

and the Klein-Gordon product, defined as

$$(\psi_{\nu sk},\psi_{\nu' s' k'})=\frac{-i\epsilon n^2}{\beta\lambda_{\nu s}^2}\int \psi_{\nu sk}\overleftrightarrow{\partial}_0\psi^*_{\nu' s' k'}rdrd\theta dz, \tag{5.28}$$

leads to

$$(\psi_{\nu sk},\psi_{\nu' s' k'})=2\pi\delta(k-k')\delta_{\nu\nu'}\delta_{ss'}. \tag{5.29}$$

The quantum-mechanical expansion for E_z becomes

$$E_z(x)=\int_{-\infty}^{\infty}\frac{dk}{2\pi}\sum_{m=1}^{\infty}\sum_s[a_{\nu sk}\psi_{\nu sk}(x)+a^\dagger_{\nu sk}\psi^*_{\nu sk}(x)], \tag{5.30}$$

with associated commutation relations

$$[a_{\nu sk},a^\dagger_{\nu' s' k'}]=2\pi\delta(k-k')\delta_{\nu\nu'}\delta_{ss'}. \tag{5.31}$$

We have to specify the continuity conditions for the fields at the transition time t_0. The component E_z will be required to be continuous,

$$E_z(x)|_{t_0^-}=E_z(x)|_{t_0^+}, \tag{5.32}$$

as well as the Klein-Gordon product,

$$\frac{-i}{\lambda_{ms}^2}\int[E_z\overleftrightarrow{\partial}_0E_z^*]_{t_0^-}rdrd\theta dz=\frac{-i}{\beta\lambda_{\nu s}^2}\int[[E_z\overleftrightarrow{\partial}_0E_z^*\ ]_{t_0^+}rdrd\theta dz, \tag{5.33}$$

from which we get

$$[\partial_0E_z(x)]_{t_0^-}=\frac{\lambda_{ms}^2}{\beta\lambda_{\nu s}^2}[\partial_0E_z(x)]_{t_0^+}. \tag{5.34}$$

The Bogoliubov transformation. We have now two kinds of basic modes, namely ψ_{msk} for $t<t_0$, and $\psi_{\nu sk}$ for $t>t_0$. There are correspondingly two vacuum states, satisfying the relations $a_{msk}|0\rangle_{msk}=0$ and $a_{\nu sk}|0\rangle_{\nu sk}=0$. As in Refs. [52,53] we may expand the modes in terms of each other,

$$\psi_{\nu sk}(x)=\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\sum_{m's'}[\gamma(\nu sk|m's'k')\psi_{m's'k'}(x)+\delta(\nu sk|m's'k')\psi^*_{m's'k'}(x)], \tag{5.35}$$

where γ and δ are the Bogoliubov coefficients [55]. The corresponding expansions for the operators are

$$a_{\nu sk}=\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\sum_{m's'}[\gamma(\nu sk|m's'k')a_{m's'k'}+\delta^*(\nu sk|m's'k')a^\dagger_{m's'k'}]. \tag{5.36}$$

It means that the average number of particles produced in the m,s,k mode per unit k space interval becomes

$$\frac{dN_{msk}}{dk}=\int_{-\infty}^{\infty}\frac{dk'}{2\pi}\sum_{m's'}|\delta(\nu sk|m's'k')|^2. \tag{5.37}$$

From Eq. (5.35) we obtain, when making use of the normalization of the scalar product corresponding to string space,

$$\delta(\nu sk|m's'k')=-(\psi_{\nu sk},\psi^*_{m's'k'})=\frac{i\epsilon n^2}{\beta\lambda_{\nu s}^2}\int\psi_{\nu sk}\overleftrightarrow{\partial}_0\psi_{m's'k'}rdrd\theta dz. \tag{5.38}$$

Here we insert expressions (5.26) and (5.7) for $\psi_{\nu sk}$ and $\psi_{m's'k'}$, and for simplicity we put $t_0=0$. Defining the quantity $I_{ss'}$ as

$$I_{ss'}=\frac{\int_0^a J_{\nu p}(\lambda_{\nu s}r)J_{mp}(\lambda_{ms'}r)rdr}{a^2|J_{\nu p+1}(\lambda_{\nu s}r)J_{mp+1}(\lambda_{ms'}r)|}, \tag{5.39}$$

we then obtain after some calculation

$$\delta(\nu sk|m's'k')=-\frac{1}{\sqrt{\beta}}\frac{\lambda_{ms}}{\lambda_{\nu s}}2\pi\delta(k+k')\delta_{mm'}\times\left[\sqrt{\frac{\omega_{\nu sk}}{\omega_{msk}}}-\sqrt{\frac{\omega_{msk}}{\omega_{\nu sk}}}\right]I_{ss'}. \tag{5.40}$$

As the value of β is very close to unity, we put $\beta=1$ everywhere except in the difference between the square roots. With $J_{\nu p}\to J_{mp}$ and $\lambda_{\nu s'}\to\lambda_{ms'}$, the numerator in Eq. (5.39) reduces to $(a^2/2)J^2_{mp+1}(\lambda_{ms}a)\delta_{ss'}$, so that approximately

$$I_{ss'}=\frac{1}{2}\delta_{ss'}. \tag{5.41}$$

Moreover, by applying the integral operator $\int dk'/2\pi$ on $[2\pi(k+k')]^2$ we obtain effectively the length L of the string. For the electromagnetic energy produced in the mode m,s,k, per unit wave-number interval, we then get

$$\frac{dW_{msk}}{dk} = \frac{\omega_{msk}}{L}\frac{dN_{msk}}{dk} = \frac{1}{4}\omega_{msk}\left(\frac{\omega_{\nu sk}}{\omega_{msk}} + \frac{\omega_{msk}}{\omega_{\nu sk}} - 2\right). \tag{5.42}$$

There are two properties of this expression worth noticing:

(1) It is independent of the opening angle α. The physical reason for this appears to be related to the fact that our region of quantization is the interior wedge region only. All the produced energy is taken to be channeled into the wedge region (we are thus not cutting out a fraction $\alpha/2\pi$ of the total produced energy). This contrasts the behavior in the cylindrically symmetric case, where the produced energy is azimuthally symmetric in the whole region $0<\theta<2\pi$ [51].

(2) The produced energy, when expressed in terms of frequencies, does not contain the refractive index n explicitly. Equation (5.42) is formally the same as Eq. (52) in Ref. [52].

We may process the expression further by making use of the asymptotic formula for the roots of the Bessel function,

$$\lambda_{ms}a = s\pi + \left(m - \frac{1}{2}\right)\frac{\pi}{2}. \tag{5.43}$$

Here it is of physical interest to consider the region around zero axial wave number, $k\approx 0$. Then $\omega_{\nu sk}\to\omega_{\nu s0}=\lambda_{\nu s}/n$, $\omega_{msk}\to\omega_{ms0}=\lambda_{ms}/n$, leading to

$$\sqrt{\frac{\omega_{\nu s0}}{\omega_{ms0}}} - \sqrt{\frac{\omega_{ms0}}{\omega_{\nu s0}}} = (\beta-1)\frac{m}{2s+m-\frac{1}{2}}, \tag{5.44}$$

where we have expanded in the small quantity $(\beta-1)$ to second order. Then,

$$\left.\frac{dW_{msk}}{dk}\right|_{k\approx 0} = \frac{\pi}{8na}(\beta-1)^2\frac{m^2}{2s+m-\frac{1}{2}}. \tag{5.45}$$

We thus see that finally the factor n turns up in the denominator; this is a characteristic property of Casimir energy expressions for dielectrics [46].

The simplest possibility $m=s=1$ yields

$$\left.\frac{dW_{11k}}{dk}\right|_{k\approx 0} = \frac{\pi}{20na}(\beta-1)^2 = \frac{4\pi}{5na}(GM)^2. \tag{5.46}$$

The total energy W produced per unit length follows by multiplying Eq. (5.46) with the wave-number width $\Delta k\sim 1/L\sim 1/a$ around $k=0$. We may take a to be of the same order as the horizon size $\sim t$, t being the time just after the Big Bang. We thus get, when leaving n unspecified,

$$W \sim \frac{1}{n}\left(\frac{GM}{t}\right)^2. \tag{5.47}$$

This is a characteristic property of cosmic string theory.

VI. CONCLUSIONS

We have computed the Casimir free energy for a wedge-shaped region bounded by perfectly conducting planes meeting in an angle. The wedge region is filled with an azimuthally symmetric material which is discontinuous at a radius a from the intersection axis. In general the wedge geometry is plagued with divergence problems. Familiar is the divergence associated with the apex, which is not relevant to the force on the circular boundary. But there are also divergences associated with the corners where the circular arc meets the wedge boundary. These divergences are manifested only in the $m=0$ modes, which possess no dependence on the angular coordinate, and have here been isolated and disregarded in the calculational part of this paper. They will not be present if the perfectly conducting boundary conditions on the wedge are replaced by periodic boundary conditions, which restrict the parameter p to be greater than unity. Then, if the speed of light is the same both inside and outside the radius a, the energy corresponding to changes in a is finite. If the speed of light differs for $r<a$ and $r>a$, the Casimir energy is finite only through second order in the discontinuity of the speed of light. These results are seen to be straightforward generalizations of results holding for dielectric/diamagnetic circular cylinders, which are recovered if $p=1$. We also consider, in the "sudden" approximation, the electromagnetic radiation produced by the appearance of a cosmic string in this geometry.

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APPENDIX A: ANALYTIC REGULARIZATION OF LOGARITHMICALLY DIVERGENT TERM

The only subtlety in the numerical calculations in Sec. IV is how the superficially logarithmically divergent terms are regulated. Starting from Eq. (4.13) we have

$$\tilde{\mathcal{E}} = \sum_{m=0}^{\infty}{}'\tilde{\mathcal{E}}_m, \tag{A1}$$

where

$$n\tilde{\mathcal{E}}_0 = \bar{\mathcal{E}}_0 - \frac{\xi^2}{4\pi a^2}\int_0^\infty dx\frac{x^5}{4(1+x^2)^3}, \tag{A2a}$$

$$n\tilde{\mathcal{E}}_m = \bar{\mathcal{E}}_m - \frac{\xi^2}{4\pi a^2}\int_0^\infty dx\frac{x^5}{4(m^2p^2+x^2)^3}. \tag{A2b}$$

Here, it will be observed that the integrals over x are logarithmically divergent. We will regulate them analytically by replacing in the numerator of both $x^5\to x^{5-s}$, where we will at the end take s to zero through positive values. Thus we have

$$2n\tilde{\mathcal{E}} - \overline{\mathcal{E}}_0 - 2\sum_{m=1}^{\infty} \overline{\mathcal{E}}_m = -\frac{\xi^2}{16\pi a^2}\int_0^\infty dx \frac{x^{5-s}}{(1+x^2)^3} \times\left(1+2\sum_{m=1}^{\infty}(mp)^{-s}\right), \tag{A3}$$

where we have let in the m terms $x=mpz$. Now the last factor is, as $s\to 0$,

$$1+2\zeta(s)p^{-s} \to -s(\ln 2\pi - \ln p), \tag{A4}$$

while the integral diverges as $s\to 0$,

$$\int_0^\infty dx \frac{x^{5-s}}{(1+x^2)^3} = \frac{1}{s}. \tag{A5}$$

Result (4.14) follows immediately. An identical argument leads to Eq. (4.27c). This argument demonstrates the importance for achieving a finite result of including both TE and TM zero modes with half weight.

APPENDIX B: CONVERGENCE CONDITION FOR ADDITIONAL ENERGY TERM ASSUMING HIGH FREQUENCY TRANSPARENCY

Energy expression (4.13),

$$\tilde{\mathcal{E}} = \frac{1}{4\pi n a^2}\sum_{m=0}^{\infty}{}' \int_0^\infty dx\, x \ln[1-\xi^2 x^2 \lambda_{mp}^2(x)], \tag{B1}$$

has an additional term consisting of one-half the $m=0$ term for the TE mode minus one-half times that of the TM mode. These modes are determined in the diaphanous case by $\Delta\tilde{\Delta}=0$, where Δ and $\tilde{\Delta}$ are the two factors in Eq. (4.8), which for the zero modes are proportional to

$$\Delta_0\tilde{\Delta}_0 \propto \left(\frac{1}{\varepsilon_1}\frac{I_0'}{I_0} - \frac{1}{\varepsilon_2}\frac{K_0'}{K_0}\right)\left(\varepsilon_1\frac{I_0'}{I_0} - \varepsilon_2\frac{K_0'}{K_0}\right). \tag{B2}$$

Then, using the Wronskian, we see that the residual zero-mode term is

$$\hat{\mathcal{E}} = -\frac{1}{16\pi n a^2}\int_0^\infty dx\, x^2 \frac{d}{dx}\ln\frac{1+\xi x\lambda_0(x)}{1-\xi x \lambda_0(x)}. \tag{B3}$$

(This just says that the reflection coefficients for the two modes are $\xi_{\rm TM}=\xi$ and $\xi_{\rm TE}=-\xi$.)

As in the perfectly conducting case, this is divergent, if ξ is constant, because

$$[I_0(x)K_0(x)]' \sim -\frac{1}{2x^2}, \quad x\to\infty, \tag{B4}$$

which means that the integral in Eq. (B3) is linearly divergent. However, if ξ is frequency dependent so that

$$\xi \sim \zeta^{-\beta}, \quad \zeta\to\infty, \tag{B5}$$

it is apparent that the integral becomes finite if $\beta > 1$.

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